Constructive toposes with countable sums as models of constructive set theory

Citation for published version:
Simpson, A & Streicher, T 2012, 'Constructive toposes with countable sums as models of constructive set theory', *Annals of Pure and Applied Logic*, vol. 163, no. 10, pp. 1419-1436.
https://doi.org/10.1016/j.apal.2012.01.013

Digital Object Identifier (DOI):
10.1016/j.apal.2012.01.013

Document Version:
Peer reviewed version

Published In:
Annals of Pure and Applied Logic

Simpson, A., & Streicher, T. (2012). Constructive toposes with countable sums as models of constructive set theory. Annals of Pure and Applied Logic, 163(10), 1419-1436doi: 10.1016/j.apal.2012.01.013

Constructive Toposes with Countable Sums as Models of Constructive Set Theory

Alex Simpson

LFCS, School of Informatics, University of Edinburgh, UK

Thomas Streicher*

Fachbereich Mathematik, Technische Universität Darmstadt, Germany

Abstract

We define a *constructive topos* to be a locally cartesian closed pretopos. The terminology is supported by the fact that constructive toposes enjoy a relationship with constructive set theory similar to the relationship between elementary toposes and (impredicative) intuitionistic set theory. This paper elaborates upon one aspect of the relationship between constructive toposes and constructive set theory. We show that any constructive topos with countable coproducts provides a model of a standard constructive set theory, $\mathbf{CZF_{Exp}}$ (that is, the variant of Aczel's Constructive Zermelo-Fraenkel set theory **CZF** obtained by weakening Subset Collection to the Exponentiation axiom). The model is constructed as a category of classes, using ideas derived from Joyal and Moerdijk's programme of *algebraic set theory*. A curiosity is that our model always validates the axiom $V = V_{\omega_1}$ (in an appropriate formulation). It follows that the full Separation schema is always refuted.

1. Introduction

The notion of elementary topos, first axiomatized by Lawvere and Tierney, provides an elegant category-theoretic abstraction of the category of sets (which we take to be axiomatized by **ZFC**). Four aspects of the relationship between elementary toposes and set theory are:

1. The category of sets is itself an elementary topos with natural numbers object (nno).
2. Elementary toposes have an internal logic which captures type-theoretic constructions on sets. This allows the objects of an elementary topos to themselves be considered as collections of unstructured elements, that is, as *abstract sets* in the sense of Lawvere [19]. However, the logic for

*Corresponding author.

manipulating such sets is a higher-order intuitionistic type theory rather than classical first-order set theory.

3. There is a natural intuitionistic first-order set theory, **BIST**, which conservatively extends higher-order intuitionistic type theory [4]. Every elementary topos with nno arises (up to equivalence) as the category of sets in a model of **BIST**. For proof-theoretic reasons, **BIST** is necessarily weaker than *Intuitionistic Zermelo-Fraenkel* set theory (**IZF**), the standard intuitionistic counterpart of **ZF** (which has the same proof-theoretic strength as **ZF**).
4. If an elementary topos has small sums (hence is cocomplete) then it models full **IZF** [14, 16].

To the constructive mathematician, unwilling to accept the impredicativity of the powersets present in both elementary toposes and set theory, the connections outlined above carry little significance. Instead, constructive mathematicians use alternative weaker formulations of set theory, in which powersets are not available. A leading such theory is Aczel's *Constructive Zermelo-Fraenkel* set theory (**CZF**). This was first presented in [1], where the main technical contribution was an interpretation of **CZF** within Martin-Löf's Intuitionistic Type Theory (**ITT**), providing a convincing demonstration of the constructive credentials of the set theory. Proof-theoretically, **CZF** has only the strength of Kripke-Platek set theory. Nevertheless, **CZF** enjoys the property that if one extends it with the Law of Excluded Middle (LEM) then one obtains classical **ZF**. Thus, in this context, LEM carries (considerable) proof-theoretic power. More to the point, as is appropriate for the constructive version of a classical theory, LEM is the only gap between constructive **CZF** and classical **ZF**.

In this paper, we take a mild variant of **CZF** as our primary set theory of interest. The theory we focus on, $\mathbf{CZF_{Exp}}$, is obtained by replacing the Subset Collection schema of **CZF** with the weaker Exponentiation Axiom, which asserts that the collection of all functions between two sets itself forms a set. (See Section 2 for a detailed presentation.) The theory $\mathbf{CZF_{Exp}}$ inherits Aczel's constructive interpretation in **ITT** from **CZF**, and still statisfies the property that its extension with LEM yield classical **ZF**. It is a natural theory in its own right, since it is the Exponentiation Axiom, not Subset Collection, that is most commonly used in the practice of constructive mathematics. Indeed, it is common for formulations of constructive set theories to take the Exponentiation Axiom as basic (for example, Myhill's **CST** [25], Friedman's systems in [15]).

It is natural to ask whether there is a compelling notion of "constructive topos", which enjoys a multi-faceted relationship with constructive set theory similar to the relationship, summarized above, between elementary toposes and impredicative intuitionistic (and classical) set theory. We argue that the appropriate notion is that of locally cartesian closed pretopos (see Section 3 for the definition). Indeed, defining *constructive topos* to mean locally cartesian closed pretopos, we have, analogously to the points above:

1. The category of sets in $\mathbf{CZF_{Exp}}$ is a constructive topos with nno.

2. Constructive toposes model an intuitionistic type theory with dependent sums, dependent products and quotients, capturing the type-theoretic constructions on sets available in $\mathbf{CZF}_{\mathbf{Exp}}$.
3. There is a natural constructive first-order set theory for which every constructive topos appears as the category of sets in a model of the theory [6]. This set theory is obtained from **BIST** by replacing the Powerset Axiom by the Exponentiation Axiom, a modification which is analogous to one possible route from **IZF** to $\mathbf{CZF}_{\mathbf{Exp}}$.
4. If a constructive topos has countable sums then it models $\mathbf{CZF}_{\mathbf{Exp}}$.

The two additional properties below further underline the naturalness of taking locally cartesian closed pretopos as the notion of constructive topos.

5. Every elementary topos is a constructive topos (but not vice versa). Thus the constructive notion of topos is a (proper) generalisation of the standard (impredicative) notion.
6. Every boolean constructive topos is a boolean elementary topos. Thus the only gap between constructive toposes and classical toposes is LEM.

Taken in combination, we believe that points 1–6 above give convincing justification for the appropriateness of taking locally cartesian closed pretopos as the definition of "constructive topos".

Points 1, 2, 5 and 6 above all describe more or less straightforward properties of locally cartesian closed pretoposes. Item 3 is covered in detail in [6]. The main technical goal of the present paper is to establish point 4: constructive toposes with countable sums model the (fairly canonical) constructive set theory $\mathbf{CZF}_{\mathbf{Exp}}$. We view this fact as an analogue for constructive toposes of the result of Fourman and Hayashi that elementary toposes with small sums model **IZF** [14, 16].

Our proof of point 4 involves a detour through models of *algebraic set theory* in the sense of Joyal and Moerdijk [18], extending the work of [4, 5, 6]. Given a constructive topos $\mathcal{E}$ with countable sums, we show that the category $\mathsf{Sh}_\infty(\mathcal{E})$ of sheaves for the countable cover topology contains within it a full subcategory of ∞-*ideals* which acts as a category of classes whose internal logic models the set theory $\mathbf{CZFA}_{\mathbf{Exp}}$ (which extends $\mathbf{CZF}_{\mathbf{Exp}}$ with a class of atoms), hence it also models $\mathbf{CZF}_{\mathbf{Exp}}$. Furthermore, up to equivalence, $\mathcal{E}$ itself is recovered as the category of sets within this category of classes. Thus the objects of an arbitrary constructive pretopos with countable sums can be seen as the collection of sets in a model of $\mathbf{CZFA}_{\mathbf{Exp}}$.

There is one perspective on this result that we wish to emphasise. The usual motivations advanced for considering weak constructive set theories, such as **CZF**, in preference to standard classical or intuitionistic set theories, such as **ZF** or **IZF**, are philosophically based, relying on scepticism over the validity of the non-constructive and impredicative principles supported by **ZF** and **IZF**. Our results supply a different and philosophically neutral reason for finding the theory $\mathbf{CZF}_{\mathbf{Exp}}$ of interest: it is has a wide range of naturally occurring models. Indeed, examples of constructive toposes with countable sums abound.

Decidable Sethood	$\mathsf{S}(x) \vee \neg\mathsf{S}(x)$
Membership	$y \in x \ \rightarrow\ \mathsf{S}(x)$
Extensionality	$\mathsf{S}(x)\ \wedge\ \mathsf{S}(y)\ \wedge\ (\forall z.\, z \in x \leftrightarrow z \in y)\ \rightarrow\ x = y$
Emptyset	$\text{Ƨ}z.\, \bot$
Pairing	$\text{Ƨ}z.\, z = x\ \vee z = y$
Equality	$\text{Ƨ}z.\, z = x\ \wedge z = y$
Union	$\text{Ƨ}z.\, \exists y \in x.\, z \in y$
(Strong) Collection	$(\forall y \in x.\, \exists z.\, \phi)\ \rightarrow$ $\exists w.\, (\,\mathsf{S}(w)\ \wedge\ (\forall y \in x. \exists z \in w.\, \phi) \wedge\ (\forall z \in w. \exists y \in x.\, \phi)\,)$
Set Induction	$(\forall x.\, (\forall y \in x.\, \phi[y])\ \rightarrow\ \phi[x])\ \rightarrow\ \forall x.\, \phi[x]$

Figure 1: Basic set-theoretic axioms

Of course, all cocomplete (hence all Grothendieck) toposes are included. But, importantly, there are naturally occurring mathematical examples of constructive toposes with countable sums that are neither cocomplete nor elementary toposes. In such examples, the stronger (impredicative) intuitionistic set theories such as **IZF** cannot be interpreted, and hence one is forced to use a weaker constructive set theory, such as $\mathbf{CZFA_{Exp}}$, if one wishes to reason with the category as if it were a category of sets.

The structure of the paper is as follows. In Section 2 we introduce the main constructive set theories of relevance to us, including **CZF**, $\mathbf{CZF_{Exp}}$ and $\mathbf{CZFA_{Exp}}$. In Section 3, we expand on the definition of constructive topos, given above, and discuss examples of constructive toposes with countable sums. In Section 4, we review the structure of categories of classes needed to provide category-theoretic models of $\mathbf{CZF_{Exp}}$ and $\mathbf{CZFA_{Exp}}$, building on work in [18, 28, 4, 6]. Our main technical contribution is presented in Section 5, where we show that the category of ∞-ideals over a constructive topos with countable sums provides a category of classes in the sense of the previous section. Finally, in Section 6, we discuss some surprising properties of the induced models of $\mathbf{CZF_{Exp}}$ and $\mathbf{CZFA_{Exp}}$. The Separation axiom always fails. More strikingly, the models validate the curious axiom $V = V_{\omega_1}$.

Throughout the paper, we use **ZFC** as the metatheory for our work. In section 7 we discuss the possibilities of weakening the metatheory.

2. Constructive set theories

The set theories in this paper are formulated in intuitionistic first-order logic with equality. Because we allow atoms, the language contains one unary predicate, S, and one binary predicate, $\in$. The formula $\mathsf{S}(x)$ expresses that x is a set. The binary predicate is set membership.

Figure 1 presents a basic set of axioms, which will be extended below. All axioms are implicitly universally quantified over their free variables. The axioms make use of the following notational devices. We write $\forall x \in y.\, \phi$ and $\exists x \in y.\, \phi$ as abbreviations for $\forall x.\,(x \in y \,\rightarrow\, \phi)$ and $\exists x.\,(x \in y \wedge \phi)$ respectively, and we refer to the prefixes $\forall x \in y$ and $\exists x \in y$ as *bounded quantifiers*. In the Set Induction schema, we use the notational device of writing $\phi[x]$ to mean a formula with the free variables x (which may or may not occur in ϕ) distinguished. Moreover, once we have distinguished x, we write $\phi[t]$ for the formula $\phi[t/x]$. Note that ϕ *is* permitted to contain free variables other than x. We also make heavy use of the notation $\text{Ƨ}x.\,\phi$, which abbreviates

$$\exists y.\,(\mathsf{S}(y) \,\wedge\, \forall x.\,(x \in y \,\leftrightarrow\, \phi))\,,$$

where y is a variable not occurring free in ϕ (cf. [3]). Thus Ƨ is generalized quantifier, where $\text{Ƨ}x.\,\phi$ reads as "there are set-many x satisfying ϕ". Using the convenience of class notation, where any formula $\phi[x]$ determines a class $\{x \mid \phi\}$, the formula $\text{Ƨ}x.\,\phi$ states that the class $\{x \mid \phi\}$ forms a set.

The first two axioms in Figure 1 are basic ontological axioms about the nature of sets and atoms. The decidability of the S predicate allows the other axioms to be formulated without making explicit assumptions that variables x are sets. For example, because of this decidability property, the Union axiom, as we have formulated it, is equivalent to its "morally correct" version:

$$\mathsf{S}(x) \wedge (\forall y \in x.\,\mathsf{S}(y)) \;\rightarrow\; \text{Ƨ}z.\,\exists y \in x.\, z \in y \;\;.$$

The proof of this exploits

$$\text{Replacement} \qquad (\forall y \in x.\,\exists! z.\,\phi) \;\rightarrow\; \text{Ƨ}z.\,\exists y \in x.\,\phi \;\;,$$

which is present as a special case of Collection. Note that our formulation allows there to be a proper class of atoms. This flexibility will be important later.

One non-standard ingredient, in the axioms of Figure 1, is the inclusion of an explicit Equality axiom. With this axiom, the schema,

$$\text{Bounded Separation} \qquad \text{Ƨ}y.\,(y \in x \,\wedge\, \phi) \qquad\qquad (\phi \text{ bounded}),$$

is derivable, where a formula is said to be *bounded* if all quantifiers in it are bounded. The proof, see [4, Section 2], again exploits Replacement.

Many constructions are naturally described using a class notation. We write U for the universal class $\{x \mid x = x\}$. Given a class $A = \{x \mid \phi\}$, we write $y \in A$ for $\phi[y]$, and we use relative quantifiers $\forall x \in A$ and $\exists x \in A$ in the obvious way. We write $A \times B$ for the product class:

$$\{p \mid \exists x \in A.\,\exists y \in B.\, p = (x, y)\} \;\;,$$

where $(x, y) = \{\{x\}, \{x, y\}\}$ is the standard Kuratowski pairing construction. Using Replacement, if A and B are both sets then so is $A \times B$ [3].

Our basic set theory is sufficient to develop Aczel's theory of inductively-defined classes. We follow the treatment in [3]. An *inductive definition* is a class Φ of ordered pairs, such that the first component of each pair in the class is a set. A class A is Φ-*closed* if, for all $(X, a) \in \Phi$, if $X \subseteq A$ then $a \in A$. The result below is proved as Theorem 5.2 of [3].

Exponentiation	Exp	$\mathsf{S}(x) \wedge \mathsf{S}(y) \;\to\; \mathsf{S}(y^x)$
(Strong) Infinity	Inf	$\text{Ƨ}x.\, x \in \mathsf{N}$
Only sets	$U = V$	$\forall x.\, \mathsf{S}(x)$.

Figure 2: Additional set-theoretic axioms

Theorem 2.1 (Class Inductive Definition Theorem)**.** *For any inductive definition Φ, there is a smallest Φ-closed class $I(\Phi)$.*

Proof. We outline the argument from [3], because some details from it will be useful to us later.

The main step is to stratify the construction of $I(\Phi)$, using elements a of the universe to indicate the strata J_a arising in the construction. Formally, $\{J_a\}_{a \in U}$ is a family of classes indexed by arbitrary sets a, so it is given by a formula $J[x, a]$, all of whose other free variables are also free in Φ. The family J_a is required to satisfy the recursive specification:

$$x \in J_a \iff \exists Y.\, (Y, x) \in \Phi \;\wedge\; \forall y \in Y.\, \exists b \in a.\, y \in J_b \;\; . \tag{1}$$

Once this is done, $I(\Phi)$ is defined as the class $\bigcup_{a \in U} J_a$. The proof that this is contained in every Φ-closed class uses Set Induction. The proof that it is itself Φ-closed uses Collection.

To finish the proof, one needs to define a family $\{J_a\}_{a \in U}$ satisfying (1). A set G of ordered pairs is called *good* if:

$$(x, a) \in G \implies \exists Y.\, (Y, x) \in \Phi \;\wedge\; \forall y \in Y.\, \exists b \in a.\, (y, b) \in G \;\; .$$

Define:

$$x \in J_a \iff \exists G.\, \mathsf{S}(G) \;\wedge\; G \subseteq U \times U \;\wedge\; G \text{ good} \;\wedge\; (x, a) \in G \;\; .$$

The proof that this satisfies the right-to-left implication of (1) again uses Collection. □

One important example of an inductively defined class is the class V of hereditary sets. This is obtained by taking, as the generating inductive definition, the class of all pairs (X, X), where X is a set. Explicitly, V is the smallest class satisfying: if $\mathsf{S}(X)$ and $X \subseteq V$ then $X \in V$.

The set theory of primary interest in this paper $\mathbf{CZFA_{Exp}}$, is obtained from our basic theory by adjoining two extra axioms. Given a set x, we write A^x for the class

$$\{f \mid \mathsf{S}(f) \;\wedge\; (\forall p \in f.\, p \in x \times A) \;\wedge\; (\forall y \in x.\, \exists! z.\, (y, z) \in f)\}$$

of all functions from x to A. We shall use standard notation for manipulating functions. Under our basic axioms, it does not follow that the class of all functions between two sets is itself a set. The Exponentiation Axiom, of Figure 2, forces this to be the case.

For the second axiom, we first define the class N of von-Neumann natural numbers, using the inductive definition consisting of all pairs of the form $(\{x\}, x \cup \{x\})$ where $\mathsf{S}(x)$, and also $(\emptyset, \emptyset)$. Thus N is the smallest class satisfying: $\emptyset \in \mathsf{N}$, and, for all $x \in \mathsf{N}$, if $\mathsf{S}(x)$ then $x \cup \{x\} \in \mathsf{N}$. The Infinity Axiom of Figure 2 states that this inductively defined class is a set. By its inductive definition, N satisfies the full induction schema:

$$\text{Induction} \qquad \phi[0] \;\wedge\; \forall x.\, (\phi[x] \to \phi[s(x)]) \;\to\; \forall x \in \mathsf{N}.\, \phi[x] \;\; ,$$

where, as usual, we write 0 for $\emptyset$, and $s(x)$ for $x \cup \{x\}$ (in the case that x is an atom, one can define $s(x)$ arbitrarily).

Figure 2 contains one other axiom, asserting that there are no atoms. This equivalently states that the equality $U = V$ holds, hence the chosen name in the figure. The two main set theories considered in this paper are:

$$\begin{aligned} \mathbf{CZFA_{Exp}} &= \text{basic axioms} + \text{Exp} + \text{Inf} \\ \mathbf{CZF_{Exp}} &= \mathbf{CZFA_{Exp}} + U{=}V \;\; . \end{aligned}$$

In this paper, we will primarily focus on the more general theory $\mathbf{CZFA_{Exp}}$. The theory $\mathbf{CZF_{Exp}}$ is easily interpretable in $\mathbf{CZFA_{Exp}}$ by relativizing all quantifiers to the class V.

To end this section, we comment on two standard set-theoretic axioms that are not theorems of $\mathbf{CZFA_{Exp}}$ (nor of $\mathbf{CZF_{Exp}}$).

Separation	Sep	$\text{Ɂ}y.\, (y \in x \,\wedge\, \phi)$	(ϕ arbitrary),
Powerset	Pow	$\text{Ɂ}x.\, \mathsf{S}(x) \wedge x \subseteq y$.	

Were classical logic assumed, the full Separation schema would follow from Replacement, and Powerset would follow from Exponentiation. However, under intuitionistic logic, neither consequence holds. As is well known, the Powerset axiom and some instances of Separation are not theorems of $\mathbf{CZF_{Exp}}$. In fact, as will be shown in Section 6, the models we construct of $\mathbf{CZFA_{Exp}}$ and $\mathbf{CZF_{Exp}}$, in Section 5, will always refute Separation, and often refute Powerset.

3. Constructive toposes

We recall some standard category-theoretic definitions. For the definitions below, let $\mathcal{C}$ be a category with finite limits.

Definition 3.1.

1. $\mathcal{C}$ is *regular* if the kernel pair $r_1, r_2 \colon R \longrightarrow A$ of every arrow $f : A \longrightarrow B$ has a coequalizer $q : B \longrightarrow C$, and regular epimorphisms are stable under pullback.
2. $\mathcal{C}$ is *exact* if it is regular and every internal equivalence relation

$$\langle r_1, r_2 \rangle \colon R \rightarrowtail A \times A$$

is a kernel pair.

3. $\mathcal{C}$ is *extensive* (also called *positive*) if it has finite coproducts, and these are disjoint and stable under pullback.
4. $\mathcal{C}$ is a *pretopos* if it is both exact and extensive.

Definition 3.2.

1. $\mathcal{C}$ has *dual images* if, for every arrow $f : C \longrightarrow D$, the inverse image map $f^{-1} : \mathrm{Sub}(D) \to \mathrm{Sub}(C)$ (where $\mathrm{Sub}(C)$ is the poset of subobjects) has a right adjoint $\forall_f \colon \mathrm{Sub}(C) \to \mathrm{Sub}(D)$ (considering f^{-1} as a functor between posets) satisfying the "Beck-Chevalley condition" of stability under pullback.
2. $\mathcal{C}$ is *locally cartesian closed* if, for every arrow $f : C \longrightarrow D$, the reindexing functor $f^* : \mathcal{C}/D \to \mathcal{C}/C$ has a right adjoint $\Pi_f \colon \mathcal{C}/D \to \mathcal{C}/C$.

It is standard that any locally cartesian closed category has dual images. Also, in a regular category, the Beck-Chevalley condition holds automatically if all right adjoints $\forall_f \colon \mathrm{Sub}(C) \to \mathrm{Sub}(D)$ exist.

We will be primarily interested in two combinations of the structure discussed above. The weaker combination of structure is sufficient for modelling first-order intuitionistic logic in the category.

Definition 3.3. An *extensive Heyting category* is an extensive regular category with dual images.

Proposition 3.4. *In every extensive Heyting category, each partial order* $\mathrm{Sub}(C)$ *of subobjects of C is a Heyting algebra. For every arrow $f \colon C \longrightarrow D$, the inverse image functor $f^{-1} : \mathrm{Sub}(D) \to \mathrm{Sub}(C)$ has both right and left adjoints $\forall_f$ and $\exists_f$ satisfying the "Beck-Chevally condition" of stability under pullbacks. In particular, $\mathcal{C}$ models intuitionistic, first-order logic with equality.*

The stronger combination of structure is the notion we are promoting as a constructive analogue the notion of topos.

Definition 3.5. A *constructive topos* is a locally cartesian closed pretopos (also called a Π-pretopos).[1]

Since a consequence of local cartesian closedness is that existing coproducts are stable, we remark that any locally cartesian closed exact category with disjoint finite coproducts is a constructive topos. In the sequel, we shall focus on a restricted class of constructive toposes, those with countable coproducts. By the previous remark, such countable coproducts are automatically stable, and their disjointness is an easy consequence of the disjointness of finite coproducts.

Obviously every constructive topos is an extensive Heyting category. Also, it is standard that every elementary topos is a constructive topos. However,

[1] Such categories are called *predicative toposes* in [6]. It has been brought to the attention of the authors that some authorities object to the word *predicative* being applied in this context. Also, the adjective *constructive* ties in better with its use in the context of constructive set theory. Other "predicative" notions of topos have been proposed elsewhere, e.g., in [24].

there are natural mathematical examples of constructive toposes that are not elementary toposes. We list five related such examples below, all given as *exact completions* of familiar categories. The first three examples arise as instances of *ex/lex completions*, that is, as exact completions as categories with finite limits; the last two are given by *ex/reg completions*, that is, as exact completions as regular categories. The reader is referred to [12, 22, 26] for detailed discussion of the different exact completion constructions and their interaction with local cartesian closedness.

1. The ex/lex completion of the category **Top** of all topological spaces.
2. The ex/lex completion of the category $\mathbf{Top_0}$ of all T_0 topological spaces.
3. The ex/lex completion of the category $\omega\mathbf{Top_0}$ of all T_0 topological space with countable base.
4. The ex/reg-completions of the categories $\mathsf{Mod}(\mathcal{P}(\omega))$ and $\mathsf{Mod}(K_2)$ of *modest sets* over the partial combinatory algebras $\mathcal{P}(\omega)$ (Scott's graph model) and K_2 (second Kleene algebra for function realizability), respectively. (See [26] for background information on realizability in categorical style. In particular, one finds there a detailed account of the partial combinatory algebras $\mathcal{P}(\omega)$ and K_2.)
5. The ex/reg-completions of the categories $\mathsf{Asm}(\mathcal{T}_{\mathcal{P}\omega})$ and $\mathsf{Asm}(\mathcal{T}_{K_2})$ of *assemblies* over the typed partial combinatory algebras $\mathcal{T}_{\mathcal{P}(\omega)}$ and $\mathcal{T}_{K_2}$ arising from the well-pointed categories $\mathsf{Mod}(\mathcal{P}\omega)$ and $\mathsf{Mod}(K_2)$, respectively. (See [20] for explanation of typed partial combinatory algebras and categories of assemblies over them.)

Here, the first two examples give rise to constructive toposes with all small sums. However, the third and the fourth example produce essentially small categories (they are equivalent to categories with $2^{2^{\aleph_0}}$ objects). The resulting constructive toposes have countable coproducts, but not all small coproducts. The fifth example produces constructive toposes which are not essentially small. They are realizability models arising from typed partial combinatory algebras generalizing the more familiar realizability toposes arising from untyped partial combinatory algebras (see [26] for a comprehensive account).

None of constructive toposes described above are toposes. In the first two cases, the categories, though locally small, are not well-powered, hence cannot be toposes. The third example is not a topos because it has no *generic proof* in the sense of Menni [22]. (Morphisms with Euclidean space $\mathbb{R}$ as codomain form at least $2^{2^{\aleph_0}}$ equivalence classes under interfactorizability. Then no countably-based T_0 space X can be the codomain of a generic proof, since there are at most $2^{\aleph_0}$ continuous maps from $\mathbb{R}$ to X.) Curiously, the category $\omega\mathbf{Top}$ of all countably-based topological spaces, does have a generic proof, and so its exact completion is a topos (it is equivalent to the realizability topos over Scott's combinatory algebra $\mathcal{P}(\omega)$). The constructive toposes of the fourth example are not toposes because there are objects with at least $2^{2^{\aleph_0}}$ many subobjects whereas all hom-sets have at most $2^{\aleph_0}$ many elements. In fact, this example subsumes the third, since the ex/reg-completion of $\mathsf{Mod}(\mathcal{P}(\omega))$ is equivalent to the ex/lex completion of $\omega\mathbf{Top}_0$. That the constructive toposes of the fifth example are not toposes has been shown in [20].

4. Categories of classes

In this section, we introduce category-theoretic models for the set theory $\mathbf{CZFA_{Exp}}$, using the approach, pioneered in Joyal and Moerdijk's *Algebraic Set Theory* [18], of axiomatizing the category-theoretic structure of the category $\mathcal{C}$ of classes. The basic idea is to axiomatize properties of a distinguished collection of "small" maps in the category, corresponding to those class functions whose fibres are sets.

Two main strands of axiomatizations have been considered in algebraic set theory. Both start by assuming basic properties of small maps, such as (S1–6) below, deriving from [18]. On top of this, one strand, originating in [18], and continuing with [7, 9, 8, 11, 10] takes, as basic, axioms asserting the exponentiability and representability of small maps, from which a powerclass functor and set-theoretic universe are derived, using assumed exactness properties of $\mathcal{C}$ and an appropriately defined well-founded tree. The second strand, developed in [28, 4, 6], requires only a regular category $\mathcal{C}$, assumes the powerclass functor and set-theoretic universe as basic, and derives the exponentiability and representability of small maps from this.

In this paper, we follow the strand of [28, 4, 6], and the reader is referred to these papers for detailed discussion and proofs of properties of the axiomatization below. Our reason for following this strand is that our construction of a particular model $\mathsf{Idl}_\infty(\mathcal{E})$ in Section 5 makes use of an explicitly defined powerclass functor and set-theoretic universe, as does the analysis of the set-theoretic properties of $\mathsf{Idl}_\infty(\mathcal{E})$ in Section 6. So it is useful to have an axiomatization based on this structure. Having said this, the notion of *countably constructive well-founded class structure* developed in this section, should be considered as a pragmatic notion designed to facilitate the proof of our main Theorem 4.6 and the properties of Section 6. Fundamentally, there is no conflict with the approach of [7, 9, 8, 11, 10], which in some ways provides a more natural category-theoretic framework for developing properties of small maps.

Let $\mathcal{C}$ be an extensive Heyting category. Let $\mathcal{S}$ be a distinguished collection of maps in $\mathcal{C}$, the *small maps*. A *small object* is an object A of $\mathcal{C}$ whose terminal map $A \longrightarrow 1$ is small. A *small relation* is a relation $r : R \rightarrowtail A \times I$ in $\mathcal{C}$ for which the second projection $\pi_2 \circ r : R \longrightarrow I$ is a small map.

The following properties of small maps are assumed as basic.

- (S1) $\mathcal{S}$ is closed under composition
- (S2) $\mathcal{S}$ is stable under pullbacks in $\mathcal{C}$
- (S3) $\mathcal{S}$ contains all regular monomorphisms of $\mathcal{C}$
- (S4) if $f \circ e$ is in $\mathcal{S}$ and e is a regular epi then f is in $\mathcal{S}$
- (S5) if $a : A \longrightarrow I$ and $b : B \longrightarrow I$ are in $\mathcal{S}$ then $[a, b] : A + B \longrightarrow I$ is also in $\mathcal{S}$.

(S6) For every small map $A \longrightarrow I$ and regular epi $X \twoheadrightarrow A$, there exists a quasi-pullback diagram[2]

$$\begin{tikzcd} B \arrow[r] \arrow[d] & X \arrow[r, two heads] & A \arrow[d] \\ J \arrow[rr, two heads] & & I \end{tikzcd} \tag{2}$$

with $J \twoheadrightarrow I$ regular epi and $B \longrightarrow J$ small.

Axiom (S6) is called the *collection axiom* in [18], since it implements the essence of set-theoretic Collection. Indeed, it asserts, in the internal logic of $\mathcal{C}$, that every cover of a small object can be refined to a small subcover, i.e., for every cover $e : X \twoheadrightarrow A$ of a small object A there exists a small object B and a map $f : B \longrightarrow X$ such that $e \circ f : B \longrightarrow A$ is still a cover. Because this assertion holds in the internal logic of $\mathcal{C}$, the object B and map f need not exist externally in $\mathcal{C}$ (but they do exist in a suitable slice of $\mathcal{C}$).

For the category $\mathcal{C}$ to have the structure of a category of classes compatible with the basic set theory of Section 2, we assume two further properties. The first states that every object A has a corresponding *powerclass* object $\mathcal{P}_s(A)$, which can intuitively be understood as the class of subsets of A.

(P) for every object A there exists an object $\mathcal{P}_s(A)$ together with a distinguished small relation $\in_A \rightarrowtail A \times \mathcal{P}_s(A)$ such that, for every small relation $r : R \rightarrowtail A \times I$, there exists a unique map $\rho : B \longrightarrow \mathcal{P}_s(A)$, called the *classifying map* for r, for which the diagram below is a pullback.

$$\begin{tikzcd} R \arrow[r] \arrow[d, tail, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & \in_A \arrow[d, tail] \\ A \times I \arrow[r, "\mathsf{id}_A \times \rho"'] & A \times \mathcal{P}_s(A) \end{tikzcd}$$

The operation $A \mapsto \mathcal{P}_s(A)$ extends to a functor on $\mathcal{C}$, whose action on morphisms maps $f \colon A \longrightarrow B$ to the classifying map for the relation r arising as the image factorization below.

$$\in_A \twoheadrightarrow R \overset{r}{\rightarrowtail} B \times \mathcal{P}_s(A) \;=\; \in_A \rightarrowtail A \times \mathcal{P}_s(A) \xrightarrow{f \times \mathsf{id}_{\mathcal{P}_s(A)}} B \times \mathcal{P}_s(A)$$

This relation is indeed small.

An important property of the structure we have identified so far is that it is *fibred*, that is, it is stable under slicing. Specifically, the small maps in any slice category $\mathcal{C}/I$ also satisfy (S1–6) and (P), and for every $f : J \longrightarrow I$, the reindexing functor $f^* : \mathcal{C}/I \to \mathcal{C}/J$ preserves the structure. Furthermore, when the map f is small, f^* has a right adjoint $\Pi_f : \mathcal{C}/J \to \mathcal{C}/I$, defining an indexed

[2] Diagram (2) is a *quasi-pullback* if it commutes and the canonical map $B \longrightarrow J \times_I A$ to the actual pullack is a regular epi.

product. This analogue of the "fundamental theorem" of topos theory is proved in [4]. Following [23], we use indexed products, to define a *polynomial functor* $Q_f \colon \mathcal{C} \to \mathcal{C}$, associated to a small map f, by:

$$X \ \mapsto \ \Sigma_{(I \longrightarrow 1)} \Pi_f \, (X \times J \xrightarrow{\pi_J} J) \ .$$

(Here Σ_f is the left adjoint to reindexing, which is given by composition.) In more readable notation, $Q_f(X) = \Sigma_{i:I} X^{f^{-1}(i)}$.

The second additional assumption we place on $\mathcal{C}$ is the existence of a set-theoretic universe in $\mathcal{C}$, freely generated from an object At of atoms by applying the functor $\mathcal{P}_S(-)$. Technically, the free generation is implemented by asking for an initial algebra for the functor $\mathsf{At} + \mathcal{P}_S(-)$.

We split the assumptions we make on the set-theoretic universe U in $\mathcal{C}$ into two parts.

(U1) There is an distinguished object At of *atoms* for which the endofunctor $\mathsf{At} + \mathcal{P}_s(-)$ has an initial algebra $[a, i] \colon \mathsf{At} + \mathcal{P}_s(U) \longrightarrow U$.

(U2) For every object A of $\mathcal{C}$ there exists a monomorphism $A \rightarrowtail U$.

Axiom (U2) says that a the set-theoretic universe U is a *universal object* in the sense of [28, 4]. As in those references, given a category $\mathcal{C}$ with collection of small maps $\mathcal{S}$ satisfying the other axioms, property (U2) can be enforced by simply cutting down $\mathcal{C}$ to its full subcategory on subobjects of U.

For the purposes of the present paper, we refer to a class $\mathcal{S}$ of small maps satisfying (S1–6), (P), and (U1–2) as *basic well-founded class structure* on an extensive Heyting category $\mathcal{C}$. A functor between categories with basic well-founded class structure is said to be *logical* if it: preserves the extensive Heyting structure, preserves small maps, preserves the powerclass structure (including the membership relations), preserves the object At, and preserves the initial algebra for the functor $\mathsf{At}+\mathcal{P}_s(-)$. Here, all preservation properties are required to hold up to isomorphism.

Proposition 4.1. *If $\mathcal{C}$ has basic well-founded class structure $\mathcal{S}$, then the small maps also provide basic well-founded class structure on every slice category $\mathcal{C}/I$. Furthermore, for every $f \colon I \longrightarrow J$, the reindexing functor $f^* \colon \mathcal{C}/J \to \mathcal{C}/I$ is logical.*

Proof. Most of the claims follow from Proposition 5.17 of [4]. For At in $\mathcal{C}/I$ we take $I^*\mathsf{At}$ which thus is preserved by f^*. It remains to show that the initial algebra of $\mathsf{At}+\mathcal{P}_s(-)$ is preserved by reindexing. For this, the argument for the closely related [8, Theorem 7.3] adapts straightforwardly to our setting. (The central idea is to show that an algebra for the functor $\mathsf{At} + \mathcal{P}_s(-)$ is initial if and only if its structure map is an isomorphism and, in addition, the algebra has no non-trivial subalgebras.) □

Following [18, 28, 4, 6], one can interpret the first-order language of Section 2 in a category $\mathcal{C}$ with basic well-founded class structure as follows. A formula $\phi(x_1,\ldots,x_k)$ is interpreted as a subobject of the object U^k, using the internal first-order logic of Heyting categories, where the interpretation of the predicates is given by: the unary predicate $\mathsf{S}(x)$ is interpreted as the subobject $i\colon \mathcal{P}_s(U) \rightarrowtail U$, where i is from the copair $[a,i]$ constituting the initial algebra in (U1); and the binary predicate $x \in y$ is interpreted as the subobject

$$\in_U \;\rightarrowtail\; U \times \mathcal{P}_s(U) \xrightarrow{\mathsf{id}_U \times i} U \times U \;\; .$$

The notion of basic well-founded class structure has been defined in such a way that each of the basic axioms of Section 2 is validated by the interpretation. Moreover, the syntactic category, as in [28, 4, 6], is an extensive Heyting category with basic well-founded class structure. Therefore a completeness result holds for interpretations of the first-order language in categories with basic well-founded class structure. We omit details, since the proof is a routine verification, along the lines of the proof in [4, Section 7].

Exploiting the connection with the basic set-theoretic axioms of Section 2, we develop an analogue, in our category-theoretic setting, of the class inductive definition theorem (Theorem 2.1). Given any $\Phi \rightarrowtail \mathcal{P}_S(X) \times X$, we say that a subobject $Y \rightarrowtail X$ is Φ-*closed* if the statement

$$\forall (u,x)\colon \Phi.\ \text{if } \forall y \in u.\ y \in Y \text{ then } x \in Y$$

holds in the internal logic of $\mathcal{E}$.

Theorem 4.2. *Suppose $\mathcal{C}$ has basic well-founded class structure $\mathcal{S}$. Given any $\Phi \rightarrowtail \mathcal{P}_S(X) \times X$, there exists a smallest Φ-closed subobject $I(\Phi) \rightarrowtail X$.*

Proof. By (U2) there is an assumed embedding $m\colon X \rightarrowtail U$. Then $\mathcal{P}_s(m)$ is also a mono [4, Proposition 5.12]. Thus, using $\mathcal{P}_s(m) \times m$ we can transfer Φ to $\Phi \rightarrowtail \mathcal{P}_s(U) \times U$ and carry out the argument there.

The argument now directly follows the proof of Theorem 2.1. In particular, the construction of the family $\{J_a\}_{a \in U}$ produces a subobject

$$\{(x,a) \mid x \in J_a\} \rightarrowtail U \times U \;\; .$$

satisfying (1) internally in $\mathcal{E}$. The required $I(\Phi) \rightarrowtail U$ is obtained as the image factorization of

$$\{(x,a) \mid x \in J_a\} \rightarrowtail U \times U \xrightarrow{\pi_1} U \;\; ,$$

where π_1 is first projection. It is easily shown that $I(\Phi) \rightarrowtail U$ factors through $m\colon X \rightarrowtail U$. That $I(\Phi)$ has the required properties as a subobject of X follows from it having these properties as a subobject of U, as in the proof of Theorem 2.1. $\square$

The next result is a useful application of Theorem 4.2, corresponding to Example 3 in Section 5.1 of [3].

Theorem 4.3. *Suppose $\mathcal{C}$ has basic well-founded class structure $\mathcal{S}$. For every small map $f\colon X \longrightarrow Y$, the polynomial functor $Q_f\colon \mathcal{C} \to \mathcal{C}$ has a fibred initial algebra.*

Proof. By proposition 6.10 of [4], by (U2) there exists $g\colon Y \to \mathcal{P}_s(U)$ fitting into a pullback square:

$$
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & Y \arrow[d, "g"] \\
\in_U \arrow[r, tail] & U \times \mathcal{P}_s(U) \arrow[r, "\pi_{\mathcal{P}_s(U)}"] & \mathcal{P}_s(U)
\end{tikzcd}
$$

Also we have $m\colon Y \rightarrowtail U$. We define the following $\Phi \rightarrowtail \mathcal{P}_s(U) \times U$.

$$\Phi \;=\; \{(u,t) : \mathcal{P}_s(U) \times U \mid \exists y : Y.\, \exists r : u^{g(y)}.\, t = (m(y), r)\} \;\; ,$$

using Kuratowski tupling in U, and the coding of individual functions $r : u^v$, for $v, u : \mathcal{P}_s(U)$ as a set of ordered pairs, hence element of U.

The carrier object of the required initial algebra is the domain of $I(\Phi) \rightarrowtail U$, given by Theorem 4.2. The algebra structure map $Q_f(I(\Phi)) \longrightarrow I(\Phi)$ sends $\langle y, r\rangle$ in $\Sigma_{y:Y}.I(\Phi)^{f^{-1}(y)}$ to $(m(y), r)$ which is in $I(\Phi)$ since $I(\Phi)$ is Φ-closed. Initiality is a consequence of $I(\Phi)$ being the smallest Φ-closed subobject. Fibredness follows from Proposition 4.1, which shows that all the structure used in the definition of $I(\Phi)$ is fibred. $\square$

We remark that Theorem 4.3 provides another contrast between our style of axiomatization and the alternative approach of [9, 8, 11, 10]. There, the property of Theorem 4.3 is assumed as an axiom (WE), which is used, together with other properties, to construct the set-theoretic universe. In this paper, we work the other way round, and derive Theorem 4.3 from an assumed set-theoretic universe.

To obtain a correspondence with $\mathbf{CZFA_{Exp}}$, we require two further axioms on the structure, implementing, and closely mirroring, the axioms Exp and Inf from Section 2. The exponentiation axiom is implemented by

(E) For every $f : J \longrightarrow I$ in $\mathcal{S}$, the functor $f^* : \mathcal{C}/I \to \mathcal{C}/J$ preserves small objects. That is, for every small $g\colon K \longrightarrow J$, the map $\Pi_f(g) \longrightarrow I$ is small.

To implement the infinity axiom, we note that, since the right injection $\mathrm{inr}\colon 1 \longrightarrow 1+1$ is small, it follows from Theorem 4.3 that the polynomial functor Q_{inr} has an initial algebra. In other words, $\mathcal{C}$ has a natural numbers object $1 + N \xrightarrow{[0,s]} N$.

(I) The natural numbers object N is a small object in $\mathcal{C}$.

We say that a class of small maps provides *constructive well-founded class structure* on an extensive Heyting category $\mathcal{C}$ if it gives basic well-founded class structure and also satisfies axioms (E) and (I).

Theorem 4.4. *The set theory* $\mathbf{CZFA_{Exp}}$ *is sound and complete relative to interpretations of the first-order language in categories with constructive well-founded class structure.*

The proof is a routine extension of the corresponding result, discussed above, relating the basic axioms of Section 2 with basic well-founded class structure.

We have now established categories with constructive well-founded class structure as an abstract framework for modelling categories of classes compatible with $\mathbf{CZFA_{Exp}}$. This framework provides a means to investigate the question:

> *Which categories can be considered as categories of sets compatible with the set theory* $\mathbf{CZFA_{Exp}}$*?*

Technically, given a category $\mathcal{C}$ with collection of small maps $\mathcal{S}$, we define its *small part* as the full subcategory $\mathcal{C}_{\mathcal{S}}$ of $\mathcal{C}$ on small objects. We then interpret the above question as: which categories arise as the small part of a category with constructive well-founded class structure? The result below gives part of the answer.

Proposition 4.5. *If* $\mathcal{S}$ *provides constructive well-founded class structure on* $\mathcal{C}$ *then* $\mathcal{C}_{\mathcal{S}}$ *is a constructive topos with natural numbers object.*

The constructive topos structure is obtained as a special case of Theorem 3.27 of [6], where a more general notion of class structure is assumed. The natural numbers object is immediate from axiom (I).

However, the "converse" of Proposition 4.5 does not hold. That is, not every constructive topos with natural numbers object arises as the small part of a category with constructive well-founded class structure. For example, there are versions of **CZF** without set induction and with natural-number-induction only for bounded formulas, such as $\mathbf{CZF}_0$ of [3], for which the associated syntactic categories of sets are nonetheless constructive toposes with natural numbers object. Since such set theories are known to be proof-theoretically weaker than $\mathbf{CZF_{Exp}}$, the resulting constructive toposes cannot arise as the small part of categories with constructive well-founded class struture. Thus, to obtain a converse to Proposition 4.5, one needs to assume further properties of a constructive topos. One possible framework for doing this might be to utilise Shulman's *stack semantics* [27], to formulate a "structural" logical property equivalent to embedability in constructive well-founded class structure, or, in logical terms, equivalent to Set Induction. As already remarked in [27], it is by no means obvious how to do this.

Instead, we take a different route and strengthen the notion of constructive well-founded class struture. As the main technical result of the paper, we characterise the categories of sets that arise as the small part of categories carrying this strengthened structure. Specifically, we assume that $\mathcal{C}$ has stable countable coproducts, and we replace axioms (S5) and (I) with a common strengthening:[3]

[3] Axiom (I) is implied because the countable copower of the terminal object 1 is a natural

(I_ω) For any countable family $(A_i \longrightarrow B)_{i\in I}$ in $\mathcal{S}$ its cotupling $\coprod_{i\in I} A_i \longrightarrow B$ is again in $\mathcal{S}$

We refer to such structure as *countably-constructive well-founded class structure*. We now state our main result, which characterises the small parts of such categories as exactly the constructive toposes with countable sums.

Theorem 4.6.

1. *If $\mathcal{C}$ is a category with countably-constructive well-founded class structure $\mathcal{S}$ then $\mathcal{C}_\mathcal{S}$ is a constructive topos with countable sums.*
2. *If $\mathcal{E}$ is a small constructive topos with countable sums then there exists a category $\mathcal{C}$ with countably-constructive well-founded class structure $\mathcal{S}$ such that $\mathcal{C}_\mathcal{S}$ is equivalent to $\mathcal{E}$.*

Statement 1 of the theorem is a straightforward consequence of Proposition 4.5, since the extra structure of countable sums is trivially transferred to $\mathcal{C}_\mathcal{S}$ from $\mathcal{C}$. The more interesting result is statement 2. For one thing, this implies that every (small) constructive topos with countable sums models $\mathbf{CZFA_{Exp}}$ (and hence $\mathbf{CZF_{Exp}}$), thus fulfilling our obligation to establish point 4 in the comparison between constructive toposes and constructive set theories of the introduction. But statement 2 goes further than this. It says that every (small) constructive topos with countable sums can itself be viewed as a category of sets compatible with the theory $\mathbf{CZFA_{Exp}}$. For this result, it seems essential to permit atoms in the theory and to allow the collection of atoms to form a class. That is, our proof does not go through if one adds the axiom $\text{Ƨ}x.\, \neg\mathsf{S}(x)$ to $\mathbf{CZFA_{Exp}}$, or equivalently the requirement that At be a small object of $\mathcal{C}$.

The one minor discrepancy between Theorem 4.6 and the result one would ideally like is the restriction to *small* constructive toposes in statement 2. This is a feature of our proof which involves constructing sheaf categories over $\mathcal{E}$. It can be circumvented, in the usual way, by using sheaves valued in an enlarged set-theoretic universe to cope with non-small $\mathcal{E}$.

5. Countable ideals

This entire section is devoted to the proof of statement 2 of Theorem 4.6 which is the main technical contribution of the paper. All categories in this section with the exception of **Set** will be assumed as small.

Our proof will be an adaptation of the proof in [6] that every constructive topos appears up to equivalence as the small part of a basic well-founded class structure satisfying axiom (E). We have to show that every constructive topos with countable sums appears up to equivalence as the small part of a countably-constructive well-founded class structure, i.e., a basic well-founded

numbers object, which is parameterized due to the stability, hence distributivity, of countable coproducts.

class structure satisfying axioms (E) and (I_ω). But for this purpose we have to recall some notions and results from [6].

First we recall Grothendieck's notion of *representable morphism* which will provide an appropriate notion of small map in various categories of interest.

Definition 5.1. Let $\mathbb{C}$ be a small category and $\mathsf{y} : \mathbb{C} \longrightarrow \widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ the Yoneda embedding. A map $f : Y \to X$ in the presheaf topos $\widehat{\mathbb{C}}$ is called *representable* iff for all $g : \mathsf{y}(A) \to X$ there exist a pullback diagram

$$\begin{tikzcd} \mathsf{y}(B) \arrow[r] \arrow[d, "\mathsf{y}(u)"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ \mathsf{y}(A) \arrow[r, "g"'] & X \end{tikzcd}$$

where $u : B \to A$ is a map in $\mathbb{C}$.

If one thinks of "small" as "representable", as first suggested by Bénabou (private communication), then representable morphisms are those families of types all of whose components are small.

Definition 5.2. A presheaf $X \in \widehat{\mathbb{C}} = \mathbf{Set}^{\mathbb{C}^{\mathsf{op}}}$ is called *separated* iff the diagonal map $\delta_X = \langle \mathsf{id}_X, \mathsf{id}_X \rangle : X \rightarrowtail X \times X$ is representable, i.e., iff for $x, y \in X(A)$ the sieve $\{u : B \to A \mid x{\cdot}u = y{\cdot}u\}$ is representable.

The following proposition from [6] ensures that every constructive topos is equivalent to the small part of some basic well-founded class structure satisfying axiom (E).

Proposition 5.3. *Let $\mathcal{E}$ be a constructive topos and $\mathsf{Sh}(\mathcal{E})$ the topos of sheaves over $\mathcal{E}$ w.r.t. the finite cover topology.*[4] *Let $\mathsf{Idl}(\mathcal{E})$ be the full subcategory of $\mathsf{Sh}(\mathcal{E})$ of separated objects. Then $\mathsf{Idl}(\mathcal{E})$ is a Heyting category with disjoint finite sums inheriting this structure from $\mathsf{Sh}(\mathcal{E})$ and the class $\mathcal{S}_\mathcal{E}$ of representable morphisms in $\mathsf{Idl}(\mathcal{E})$ gives rise to a basic well-founded class structure on $\mathsf{Idl}(\mathcal{E})$ satisfying axiom* (E)*.*

The sum $\mathsf{At}_\mathcal{E} = \coprod\limits_{A \in \mathsf{Ob}(\mathcal{E})} \mathsf{y}(A)$ in $\widehat{\mathbb{C}}$ is an object of $\mathsf{Idl}(\mathcal{E})$. In $\mathsf{Idl}(\mathcal{E})$ there exists an initial algebra $U_\mathcal{E}$ of the endofunctor $\mathsf{At}_\mathcal{E} + \mathcal{P}_s(-)$ on $\mathsf{Idl}(\mathcal{E})$.

We recall that by Yoneda the small power object $\mathcal{P}_s(X)$ in $\mathsf{Idl}(\mathcal{E})$ is given by

$$\mathcal{P}_s(X)(A) \cong \{R \rightarrowtail X \times \mathsf{y}(A) \mid R \text{ representable}\}$$

since a relation $r : R \rightarrowtail X \times \mathsf{y}(A)$ is small iff $\pi_2 \circ r : R \to \mathsf{y}(A)$ is in $\mathcal{S}_\mathcal{E}$ iff R is representable.

For later reference we also recall the following characterisation of separated objects in $\mathsf{Sh}(\mathcal{E})$ from [5, 6] originally suggested by A. Joyal.

[4] which is generated by finite jointly epic families in $\mathcal{E}$, see [17] where it is called "coherent" topology

Proposition 5.4. *For $X \in \mathsf{Sh}(\mathcal{E})$ the following conditions are equivalent*

(1) $X \in \mathsf{Idl}(\mathcal{E})$
(2) *for every $f : \mathsf{y}(A) \longrightarrow X$ its image in $\mathsf{Sh}(\mathcal{E})$ is representable*
(3) *X arises as colimit in $\widehat{\mathcal{E}} = \mathbf{Set}^{\mathcal{E}^{\mathsf{op}}}$ of some directed diagram $D : \mathcal{I} \longrightarrow \mathsf{Sh}(\mathcal{E})$ where all $D(i)$ are representable and all $D(i \leq j)$ are monic.*

Directed colimits of monos of representables are called *ideal colimits*. It follows from Proposition 5.4 that $\mathsf{Idl}(\mathcal{E})$ is closed under ideal colimits and those are computed as in $\widehat{\mathcal{E}}$.

This finishes our recap of the relevant results from [6] and we now turn to the main goal of this section.

Let $\mathcal{E}$ be a constructive topos with countable sums. We consider $\mathcal{E}$ as endowed with the *countable cover* topology where a sieve S *covers* I if S contains a countable jointly epic family of morphisms. This is a Grothendieck topology because countable sums are stable. We write $\mathsf{Sh}_\infty(\mathcal{E})$ for the category of sheaves on $\mathcal{E}$ w.r.t. the countable cover topology. Since any coherent cover is in particular a countable cover the category $\mathsf{Sh}_\infty(\mathcal{E})$ is a full subcategory of $\mathsf{Sh}(\mathcal{E})$ and the inclusion has a finite limit preserving left adjoint a called *associated sheaf* functor.

Obviously all representable objects are sheaves w.r.t. the countable cover topology and thus *a fortiori* w.r.t. the finite cover topology. Notice that the Yoneda functor $\mathsf{y} : \mathcal{E} \longrightarrow \mathsf{Sh}_\infty(\mathcal{E})$ preserves countable sums because for a countable family (A_i) in $\mathcal{E}$ its colimiting cone $(\mathsf{in}_i : A_i \to \coprod A_i)$ generates a countable cover. Thus, in particular, for $N = \coprod\limits_{n\in\omega} 1_{\mathcal{E}}$ we have $\mathsf{y}(N) \cong \coprod\limits_{n\in\omega} \mathsf{y}(1_{\mathcal{E}}) \cong \coprod\limits_{n\in\omega} 1_{\mathsf{Sh}_\infty(\mathcal{E})}$ and, accordingly, $\mathsf{y}(N)$ is a natural numbers object in $\mathsf{Sh}_\infty(\mathcal{E})$.

Next we show that $\mathsf{Sh}_\infty(\mathcal{E})$ is closed under a particular kind of colimit in $\widehat{\mathcal{E}}$., which we call "∞-ideal colimits".

Definition 5.5. A poset $\mathcal{I}$ is ω_1*-directed* iff every countable subset of $\mathcal{I}$ has an upper bound in $\mathcal{I}$. An ∞-ideal diagram in a category $\mathcal{C}$ is a mono preserving functor $D : \mathcal{I} \longrightarrow \mathcal{C}$ for some ω_1-directed poset $\mathcal{I}$ considered as a category. An ∞*-ideal colimit* is a colimit of an ∞-ideal diagram.

The following basic fact will turn out as crucial.

Proposition 5.6. *The category $\mathsf{Sh}_\infty(\mathcal{E})$ is closed under ∞-ideal colimits taken in $\widehat{\mathcal{E}}$.*

Proof. Suppose $D : \mathcal{I} \longrightarrow \mathsf{Sh}_\infty(\mathcal{E})$ is an ∞-ideal diagram in $\widehat{\mathcal{E}}$ and let D_∞ be its colimit in $\widehat{\mathcal{E}}$. Notice that all maps of the colimiting cone $(\mathrm{in}_i : D_i \longrightarrow D_\infty)$ are monic and that every element in a fibre of D_∞ appears already in the image of some in_i. For sake of simplicity, we may pretend that all in_i are inclusions in the sense that all their components are subset incluions.

We have to show that D_∞ is a sheaf w.r.t. the countable cover topology. For this purpose suppose $(f_\alpha : A_\alpha \longrightarrow A)_{\alpha\in I}$ is a countable jointly epic family in $\mathcal{E}$ and (d_α) is a compatible family of elements of $D_\infty(A_\alpha)$. Thus, by the considerations in the previous paragraph and since $\mathcal{I}$ is ω_1-directed there exists an $i \in \mathcal{I}$ s.t. all d_α show up already in the image of in_i. Since D_i is a sheaf w.r.t. the countable cover topology there exists a unique $d \in D_i(A)$ with $d_\alpha = D_i(f_\alpha)(d)$ for all $\alpha \in I$. Thus, by the considerations in the previous paragraph this d (considered as an element of D_∞) is the unique one satisfying $d_\alpha = D_\infty(f_\alpha)(d)$ for all $\alpha \in I$. $\square$

Moreover, in analogy with Prop. 5.4 we have the following characterisation of separated objects in $\mathsf{Sh}_\infty(\mathcal{E})$.

Proposition 5.7. *For $X \in \mathsf{Sh}_\infty(\mathcal{E})$ the following conditions are equivalent*

(1) *$X \in \mathsf{Idl}(\mathcal{E})$, i.e. is separated*
(2) *for every $f : \mathsf{y}(A) \longrightarrow X$ its image in $\mathsf{Sh}_\infty(\mathcal{E})$ is representable*
(3) *X arises as an ∞-ideal colimit of representable objects in $\widehat{\mathcal{E}}$.*

Proof. For showing that (1) implies (2) suppose X is separated and $f : \mathsf{y}(A) \to X$. By Proposition 5.4 the image of f in $\mathsf{Sh}(\mathcal{E})$ is representable. Since the associated sheaf functor $\mathsf{a} : \mathsf{Sh}(\mathcal{E}) \to \mathsf{Sh}_\infty(\mathcal{E})$ preserves monos, (regular) epis and representable objects the image of $f : \mathsf{y}(A) \to X$ in $\mathsf{Sh}_\infty(\mathcal{E})$ coincides with its image in $\mathsf{Sh}(\mathcal{E})$ (which can be seen by applying a to the epi-mono factorisation of f in $\mathsf{Sh}(\mathcal{E})$. Thus, the image of f in $\mathsf{Sh}_\infty(\mathcal{E})$ is representable.

For showing that (2) implies (3) suppose that for every $f : \mathsf{y}(A) \longrightarrow X$ its image in $\mathsf{Sh}_\infty(\mathcal{E})$ is representable. Thus, in $\widehat{\mathcal{E}}$ the object X is the colimit of its representable subobjects in $\mathsf{Sh}_\infty(\mathcal{E})$. Since the associated sheaf functor $\mathsf{a} : \mathsf{Sh}(\mathcal{E}) \longrightarrow \mathsf{Sh}_\infty(\mathcal{E})$ preserves monos, (regular) epis, (countable) sums and representable objects and the latter are closed under countable sums in $\mathsf{Sh}_\infty(\mathcal{E})$ the subobjects of X in $\mathsf{Sh}_\infty(\mathcal{E})$ give rise to an ∞-ideal diagram in $\widehat{\mathcal{E}}$. Thus X is an ∞-ideal colimit of representable objects in $\widehat{\mathcal{E}}$.

That (3) implies (1) follows from the respective implication in Proposition 5.4 since ∞-ideal colimits are in particular also ideal colimits. $\square$

For constructing appropriate models of $\mathbf{CZF_{Exp}}$ from $\mathcal{E}$ we consider the following subcategory of $\mathsf{Sh}_\infty(\mathcal{E})$.

Definition 5.8. A *countable ideal* in $\mathcal{E}$ is a separated object of $\mathsf{Sh}_\infty(\mathcal{E})$. We write $\mathsf{Idl}_\infty(\mathcal{E})$ for the full subcategory of $\mathsf{Sh}_\infty(\mathcal{E})$ on countable ideals in $\mathcal{E}$.

From Proposition 5.7 it follows that $\mathsf{Idl}_\infty(\mathcal{E}) = \mathsf{Sh}_\infty(\mathcal{E}) \cap \mathsf{Idl}(\mathcal{E})$.

Proposition 5.9. *The category $\mathsf{Idl}_\infty(\mathcal{E})$ is closed under ∞-ideal colimits taken in $\widehat{\mathcal{E}}$.*

Proof. The category $\mathsf{Idl}(\mathcal{E})$ is closed under ideal colimits taken in $\widehat{\mathcal{E}}$. Since ∞-ideal diagrams are in particular also ideal diagrams the category $\mathsf{Idl}(\mathcal{E})$ is closed under ∞-ideal colimits taken in $\widehat{\mathcal{E}}$. From this together with Proposition 5.6 it follows that $\mathsf{Idl}_\infty(\mathcal{E}) = \mathsf{Sh}_\infty(\mathcal{E}) \cap \mathsf{Idl}(\mathcal{E})$ is closed under ∞-ideal colimits taken in $\widehat{\mathcal{E}}$. $\square$

The following lemma will be crucial for verifying that the class of representable morphisms in $\mathsf{Idl}_\infty(\mathcal{E})$ gives rise to a countably-constructive well-founded class structure.

Lemma 5.10. *The adjunction* $\mathsf{a} \dashv \mathsf{i} : \mathsf{Sh}(\mathcal{E}) \hookrightarrow \mathsf{Sh}_\infty(\mathcal{E})$ *restricts to an adjunction* $\mathsf{a} \dashv \mathsf{i} : \mathsf{Idl}(\mathcal{E}) \hookrightarrow \mathsf{Idl}_\infty(\mathcal{E})$ *which is a localisation, i.e. the left adjoint preserves finite limits. The category* $\mathsf{Idl}_\infty(\mathcal{E})$ *is regular and an object* $A \in \mathsf{Idl}(\mathcal{E})$ *is in* $\mathsf{Idl}_\infty(\mathcal{E})$ *iff* $\mathsf{Idl}(\mathcal{E})(m, A)$ *is a bijection for all monos* m *in* $\mathsf{Idl}(\mathcal{E})$ *that are mapped to isomorphisms by* a*.*

Proof. The associated sheaf functor $\mathsf{a} \dashv \mathsf{i} : \mathsf{Sh}_\infty(\mathcal{E}) \hookrightarrow \mathsf{Sh}(\mathcal{E})$ preserves colimits, finite limits and representable objects. Thus, it also preserves separated objects and, accordingly, the functor a restricts to a functor from $\mathsf{Idl}(\mathcal{E})$ to $\mathsf{Idl}_\infty(\mathcal{E})$ left adjoint to the inclusion $\mathsf{i} : \mathsf{Idl}_\infty(\mathcal{E}) \hookrightarrow \mathsf{Idl}(\mathcal{E})$. The functor $\mathsf{a} : \mathsf{Idl}(\mathcal{E}) \longrightarrow \mathsf{Sh}_\infty(\mathcal{E})$ preserves finite limits since $\mathsf{Idl}(\mathcal{E})$ and $\mathsf{Idl}_\infty(\mathcal{E})$ inherit finite limits from $\mathsf{Sh}(\mathcal{E})$ and $\mathsf{Sh}_\infty(\mathcal{E})$, respectively. Thus, the adjunction $\mathsf{a} \dashv \mathsf{i} : \mathsf{Idl}(\mathcal{E}) \hookrightarrow \mathsf{Idl}_\infty(\mathcal{E})$ is a localisation. Thus, since $\mathsf{Idl}(\mathcal{E})$ is regular and regular categories are closed under localisation the category $\mathsf{Idl}_\infty(\mathcal{E})$ is also regular. It follows from Proposition 5.6.4 of vol.1 of [13] that an object $A \in \mathsf{Idl}(\mathcal{E})$ is in $\mathsf{Idl}_\infty(\mathcal{E})$ iff $\mathsf{Idl}(\mathcal{E})(m, A)$ is a bijection for all monos m in $\mathsf{Idl}(\mathcal{E})$ that are inverted by a. □

Proposition 5.11. $\mathsf{Idl}_\infty(\mathcal{E})$ *is a Heyting category with stable and disjoint sums.*

Proof. $\mathsf{Idl}_\infty(\mathcal{E})$ is a Heyting category since by Proposition 5.3 $\mathsf{Idl}(\mathcal{E})$ is a Heyting category and this property is stable under localisation.

Notice that in $\widehat{\mathcal{E}}$ separated objects are closed under small sums. Thus, since $\mathsf{a} : \widehat{\mathcal{E}} \longrightarrow \mathsf{Sh}_\infty(\mathcal{E})$ is a left adjoint preserving finite limits and representable objects it follows that separated objects in $\mathsf{Sh}_\infty(\mathcal{E})$ are closed under small sums in $\mathsf{Sh}_\infty(\mathcal{E})$ which are stable and disjoint since $\mathsf{Sh}_\infty(\mathcal{E})$ is a Grothendieck topos and the initial object of $\mathsf{Sh}_\infty(\mathcal{E})$ is separated. □

Proposition 5.12. *The class* $\mathcal{S}_\mathcal{E}$ *of representable morphisms in* $\mathsf{Idl}_\infty(\mathcal{E})$ *is a class of small maps, i.e. validates the axioms* (S1)-(S6)*.*

Proof. Since $\mathsf{a} : \mathsf{Sh}(\mathcal{E}) \longrightarrow \mathsf{Sh}_\infty(\mathcal{E})$ is a left adjoint preserving finite limits and representable objects it preserves representable morphism. By Lemma 5.10 the functor a sends also $\mathsf{Idl}(\mathcal{E})$ to $\mathsf{Idl}_\infty(\mathcal{E})$. Thus a sends representable morphisms in $\mathsf{Idl}(\mathcal{E})$ to $\mathcal{S}_\mathcal{E}$. W.l.o.g. we may assume that a is the identity on $\mathsf{Sh}_\infty(\mathcal{E})$. It is now easy to verify that $\mathcal{S}_\mathcal{E}$ is a class of small maps in $\mathsf{Idl}_\infty(\mathcal{E})$. We give the arguments for (S2) and (S6) and leave the routine verification of the remaining conditions to the reader.

For (S2) suppose $f : Y \longrightarrow X$ is in $\mathcal{S}_\mathcal{E}$ and $g : Z \longrightarrow X$ is in $\mathsf{Idl}_\infty(\mathcal{E})$.

Now consider the pullback

$$
\begin{tikzcd}
U \arrow[r, "q"] \arrow[d, "g^* f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
Z \arrow[r, "f"'] & X
\end{tikzcd}
$$

in $\mathsf{Sh}_\infty(\mathcal{E})$ which is inherited from $\widehat{\mathcal{E}}$. Thus, since representable morphisms are stable under pullbacks in $\widehat{\mathcal{E}}$ it follows that $g^* f$ is representable and thus in $\mathcal{S}_{\mathcal{E}}$.

In order to verify condition (S6) suppose $Y \longrightarrow X$ is in $\mathcal{S}_{\mathcal{E}}$ and $Z \twoheadrightarrow Y$ is a regular epi in $\mathsf{Idl}_\infty(\mathcal{E})$. Using collection in **Set** we can fit these two maps into a quasi-pullback diagram

$$
\begin{tikzcd}
V \arrow[r] \arrow[d] & Z \arrow[r, two heads] & Y \arrow[d] \\
U \arrow[rr, two heads] & & X
\end{tikzcd}
$$

in $\widehat{\mathcal{E}}$ where $U \longrightarrow X$ is a regular epi and $V \longrightarrow U$ is representable. Then applying the associated sheaf functor a to it we obtain a quasi-pullback

$$
\begin{tikzcd}
\mathsf{a}(V) \arrow[r] \arrow[d] & Z \arrow[r, two heads] & Y \arrow[d] \\
\mathsf{a}(U) \arrow[rr, two heads] & & X
\end{tikzcd}
$$

in $\mathsf{Sh}_\infty(\mathcal{E})$ whose left side is representable. □

Using Proposition 5.7 and results from [6] one can show that

Proposition 5.13. *The functor $\mathcal{P}_s : \mathsf{Idl}(\mathcal{E}) \longrightarrow \mathsf{Idl}(\mathcal{E})$ preserves ideal and ∞-ideal colimits. Moreover, it preserves $\mathsf{Idl}_\infty(\mathcal{E})$ and thus restricts to a functor $\mathcal{P}_s : \mathsf{Idl}_\infty(\mathcal{E}) \longrightarrow \mathsf{Idl}_\infty(\mathcal{E})$ which preserves ∞-ideal colimits.*

Proof. From [6] it follows that $\mathcal{P}_s$ commutes with ideal colimits and thus with ∞-ideal colimits and that $\mathcal{P}_s$ preserves separatedness. Thus it suffices to show that $\mathcal{P}_s(A) \in \mathsf{Sh}_\infty(\mathcal{E})$ for every $A \in \mathcal{E}$.

For this purpose suppose $(u_n : I_n \longrightarrow I)$ is a countable cover of I and $\big(S_n \in \mathcal{P}_s(A)(I_n)\big)$ is a family compatible in the sense that whenever $u_n v = u_m w$ for some arrows v and w with source J then $(v \times A)^* S_n \cong (w \times A)^* S_m$ as subobjects of $J \times A$. Then, due to the assumptions on $\mathcal{E}$ the subobject $S = \bigvee (u_n \times A)[S_n]$ of $I \times A$ is the unique $S \in \mathcal{P}_s(A)(I)$ with $S \cdot u_n \cong S_n$ for all n. □

Proposition 5.14. *Representable morphisms in $\mathsf{Idl}_\infty(\mathcal{E})$ validate axiom* (E).

Proof. In Proposition 4.26 of [6] it has been shown that $\mathsf{Idl}(\mathcal{E})$ validates axiom (E). Since $\mathsf{Idl}_\infty(\mathcal{E})$ appears as localisation of $\mathsf{Idl}(\mathcal{E})$ property (E) is preserved because by a standard argument the inclusion of $\mathsf{Idl}_\infty(\mathcal{E})$ into $\mathsf{Idl}(\mathcal{E})$ preserves dependent products. □

Proposition 5.15. *Representable morphisms in* $\mathsf{Idl}_\infty(\mathcal{E})$ *validate axiom* (I_ω).

Proof. Suppose $(f_i : Y_i \longrightarrow X)_{i\in I}$ is a countable family of representable morphisms in $\mathsf{Idl}_\infty(\mathcal{E})$. Let $f : \coprod_{i\in I} Y_i \to X$ be the source tupling of the f_i. Suppose $g : \mathsf{y}(A) \longrightarrow X$ and $\mathsf{y}(h_i) = g^* f_i : \mathsf{y}(B_i) \longrightarrow \mathsf{y}(A)$ for $i \in I$. Then $g^* f$ is isomorphic to the source tupling $h : \coprod_{i\in I} \mathsf{y}(B_i) \to \mathsf{y}(A)$ of the h_i. But since y preserves countable sums the source of $g^* f$ is isomorphic to $\mathsf{y}(\coprod_{i\in I} B_i)$, i.e. representable. Thus f is a representable morphism as desired. □

Summarising these results we observe that

Theorem 5.16. *The representable morphisms give rise to a basic well-founded class structure on* $\mathsf{Idl}_\infty(\mathcal{E})$ *validating axioms* (E) *and* (I_ω).

Proof. Immediate from Propositions 5.11, 5.12, 5.13, 5.14 and 5.15. □

Now we turn to the construction of universes.

Proposition 5.17. *For every object* A *of* $\mathsf{Idl}_\infty(\mathcal{E})$ *the functor* $A+\mathcal{P}_s(-)$ *has a initial algebra* $U_A \cong A + \mathcal{P}_s(U_A)$.

Proof. In [6] it has been shown that $\mathcal{P}_s$ preserves monos and thus the functor $F_A = A+\mathcal{P}_s(-)$ also preserves monos. Since $A+(-)$ preserves univers directed colimits it follows from Proposition 5.13 that F_A preserves ∞-ideal colimits.

Consider now the ∞-ideal diagram $\left(F_A^\alpha(0)\right)_{\alpha<\omega_1}$ in $\mathsf{Idl}_\infty(\mathcal{E})$ where $F_A^{\alpha+1}(0) = F_A(F_A^\alpha(0))$ and $F_A^\lambda(0) = \mathsf{a}\,(\mathrm{colim}_{\alpha<\lambda} F_A^\alpha(0))$ for limit ordinals $\lambda < \omega_1$. By Proposition 5.9 the ∞-ideal colimit $U_A = \mathrm{colim}_{\alpha<\omega_1} F_A^\alpha(0)$ exists in $\mathsf{Idl}_\infty(\mathcal{E})$. Since by Proposition 5.13 the functor F_A preserves ∞-ideal colimits in $\mathsf{Idl}_\infty(\mathcal{E})$ it is straightforward and well-known that $U_A = \mathrm{colim}_{\alpha<\lambda} F_A^\alpha(0)$ carries the structure of an initial F_A-algebra in $\mathsf{Idl}\infty(\mathcal{E})$. □

Due to Proposition 5.11 the sum $\mathsf{At}_\mathcal{E} = \coprod_{A\in\mathsf{Ob}(\mathcal{E})} \mathsf{y}(A)$ exists in $\mathsf{Idl}_\infty(\mathcal{E})$. Thus, by Proposition 5.17 there exists an initial fixpoint $U_\mathcal{E} \cong \mathsf{At}_\mathcal{E} + \mathcal{P}_s(U_\mathcal{E})$.

Theorem 5.18. *Let* $\mathcal{C}_\mathcal{E}$ *be the full subcategory of* $\mathsf{Idl}_\infty(\mathcal{E})$ *on subobjects of* $U_\mathcal{E}$ *and* $\mathcal{S}_\mathcal{E}$ *the class of representable morphisms in* $\mathcal{C}_\mathcal{E}$. *Then* $(\mathcal{C}_\mathcal{E}, \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ *is a countably-constructive well-founded class structure whose small part is equivalent to* $\mathcal{E}$.

Proof. It is easy to straightforward to check that Propositions 5.11, 5.12, 5.13, 5.14, 5.15 and 5.17 restrict to $\mathcal{C}_\mathcal{E}$. Thus $(\mathcal{C}_\mathcal{E}, \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ is a countably-constructive well-founded class structure.

Since for every object A of $\mathcal{E}$ we have $\mathsf{y}(A) \rightarrowtail \mathsf{At}_\mathcal{E} \rightarrowtail U_\mathcal{E}$ the small part of $\mathcal{C}_\mathcal{E}, \mathcal{S}_\mathcal{E}, U_\mathcal{E})$ is equivalent to $\mathcal{E}$. □

Thus we have finally proved statement 2 of Theorem 4.6. Notice that the object $\mathsf{At}_\mathcal{E}$ in $\mathsf{Idl}_\infty(\mathcal{E})$ cannot be small as otherwise one could derive an analogue of Russell's paradox. However, we need such a big object $\mathsf{At}_\mathcal{E}$ for obtaining a countably-constructive well-founded class structure whose small part is equivalent to $\mathcal{E}$.

6. Properties of the model

Throughout this section, let $\mathcal{E}$ be a small constructive topos with countable sums. We investigate properties of the model $\mathsf{Idl}_\infty(\mathcal{E})$ of $\mathbf{CZFA_{Exp}}$.

6.1. The axioms $U = U_{\omega_1}$ and $V = V_{\omega_1}$

In this subsection, we show that our model $\mathsf{Idl}_\infty(\mathcal{E})$ validates the axiom $U = U_{\omega_1}$ (and hence $V = V_{\omega_1}$), meaning roughly that U is constructed by ω_1 iterations of the powerclass operation. Although this property is rather blatantly built into the consruction of the universe as an ω_1-colimit in $\mathsf{Idl}_\infty(\mathcal{E})$, from a set-theoretic perspective, $U = U_{\omega_1}$ is a surprising and somewhat pathological axiom. Moreover, there are subtleties in formulating this axiom in $\mathbf{CZFA_{Exp}}$. As we shall see, there are two candidates for the ordinal ω_1.

Working in $\mathbf{CZFA_{Exp}}$, we stratify the universe U according to an index a indicating the current level in the universe, using the construction in the proof of Theorem 2.1. Specifically, we define sublasses U_a of U satisfying the recursive specification

$$U_a \;=\; \bigcup_{b\in a} \{x \mid \forall y \in x.\, y \in U_b\} \;\; .$$

Note that U_a is thus defined as a set-indexed union of classes. In the case that a is an atom of $\emptyset$, we have that U_a is empty. However, when a is an inhabited set, $\emptyset$ belongs to U_a, as does every atom in the universe. The construction can be adapted to stratify V by

$$V_a \;=\; \bigcup_{b\in a} \{x \mid \mathsf{S}(x) \,\wedge\, \forall y \in x.\, y \in V_b\} \;=\; V \cap U_a \;\; .$$

By simple applications of Set Induction, one has:

$$U \;=\; \bigcup_a U_a \qquad\qquad V \;=\; \bigcup_a V_a \;\; .$$

We also extend the indices to classes A, defining

$$U_A \;=\; \bigcup_{a\in A} U_a \qquad\qquad V_A \;=\; \bigcup_{a\in A} V_a \;\; .$$

We remark on one feature that distinguishes the above constructions from their analogues in classical set theory. In $\mathbf{CZFA_{Exp}}$, the "second level" $V_{\{\emptyset,\{\emptyset\}\}}$ is a set if and only if the Powerset axiom holds. Thus, in general, V_a may be a proper class, even for $a = \{\emptyset, \{\emptyset\}\}$.

Although we have thus far allowed the indices to be arbitrary, in the cases of interest to us, the indices will be ordinal classes. As usual, a *transitive class* is a class A for which $y \in x \in A$ implies $y \in A$. An *ordinal class* is a transitive class all of whose elements are transitive sets. An *ordinal* is an ordinal class that is a set. Note that, although a general transitive class may contain atoms, an ordinal class cannot. Indeed, every element of an ordinal class is an ordinal.

For an ordinal α and ordinal class β we write $\alpha < \beta$ to mean $\alpha \in \beta$. Also, for two ordinal classes α, β, we write $\alpha \leq \beta$ to mean $\alpha \subseteq \beta$.

Next, we define ordinal classes corresponding to the first uncountable ordinal ω_1 in classical set theory. We give two definitions. The first is the natural one. Define $\omega_1^\sharp$ to be the smallest class that: contains $\emptyset$, is closed under ordinal successor $x \mapsto x \cup \{x\}$, and is closed under N-indexed unions. (This can be coded up using Theorem 2.1, using an inductive definition, Φ, containing the pairs: $(\emptyset, \emptyset)$; $(\{x\}, x \cup \{x\})$, for every set x; and $(\{f(n) \mid n \in \mathsf{N}\}, \bigcup_{n \in \mathsf{N}} f(n))$, for every function $f \colon \mathsf{N} \to \{x \mid \mathsf{S}(x)\}$.) Since the class of ordinals is closed under the specified operations, the class $\omega_1^\sharp$ consists of ordinals. Moreover, the subclass of $\omega_1^\sharp$ consisting of those elements whose transitive closures are subsets of $\omega_1^\sharp$ is also closed under the operations. Therefore, this subclass is the whole of $\omega_1^\sharp$. That is, $\omega_1^\sharp$ is transitive. Thus $\omega_1^\sharp$ is indeed an ordinal class.

The second definition is indirect. We first define Brouwer's second number class, W_1, as the absolutely free algebra generated by: one constant, 0; one unary operation, s; and one N-ary operation, l. (Again, this can be obtained from using Theorem 2.1, using the inductive definition, Φ, containing the pairs: $(\emptyset, (0, \emptyset))$; $(\{x\}, (s, x))$, for every $x \in U$; $(\{f(n) \mid n \in \mathsf{N}\}, (l, f))$, for every $f \colon \mathsf{N} \to U$; where $0, s, l$ are three chosen distinct elements of U.) Interpreting 0 as $\emptyset$, the operation s as $x \mapsto x \cup \{x\}$, and the operation l as countable union, the class of ordinals is an algebra for the signature. Therefore there is a unique algebra homomorphism h from W_1 to the class of ordinals. Define $\omega_1^\flat$ to be the image of h. Consider the subclass of W_1 consisting of those elements t for which the transitive closure of $h(t)$ is a subset of $\omega_1^\flat$. This subclass contains 0 and is easily shown to be closed under s and l, hence is the whole of W_1. Thus $\omega_1^\flat$ is transitive, whence an ordinal class. Also, $\omega_1^\flat \leq \omega_1^\sharp$, because the image of the homomorphism restricts to $\omega_1^\sharp$, since this too is closed under the operations. If countable choice is assumed then it is easy to show that $\omega_1^\flat = \omega_1^\sharp$, but it does not seem possible to prove this in $\mathbf{CZFA_{Exp}}$. (It would be interesting to have a countermodel.)

Theorem 6.1. $\mathsf{Idl}_\infty(\mathcal{E}) \models U = U_{\omega_1^\flat}$*, hence* $\mathsf{Idl}_\infty(\mathcal{E}) \models V = V_{\omega_1^\flat}$.

Because $\omega_1^\flat \leq \omega_1^\sharp$, we have $U_{\omega_1^\flat} \subseteq U_{\omega_1^\sharp}$, and so also $\mathsf{Idl}_\infty(\mathcal{E}) \models U = U_{\omega_1^\sharp}$, and similarly $\mathsf{Idl}_\infty(\mathcal{E}) \models V = V_{\omega_1^\sharp}$. The theorem is stated for the ordinal class $\omega_1^\flat$ because this is the stronger property.

Proof. As in the proof of Theorem 4.2, the internalization in $\mathcal{E}$ of the set-theoretic definition of the stratification U_a of U defines an object

$$\{(x, a) \mid x \in U_a\} \rightarrowtail U_\mathcal{E} \times U_\mathcal{E} \xrightarrow{\pi_2} U_\mathcal{E} \tag{3}$$

of the slice category $\mathcal{E}/U_\mathcal{E}$. Let W_1 in $\mathcal{E}$ be the initial algebra, easily given by Theorem 4.3, for one constant 0, one unary operation s, and one N-ary operation l. Let $h \colon W_1 \longrightarrow U_\mathcal{E}$ be the homomorphism given by the initial algebra property with respect to the algebra structure on $U_\mathcal{E}$ that interprets

0 as $\emptyset$, the operation s as $x \mapsto x \cup \{x\}$, and l as countable union (as above). Pulling back (3) along h, we obtain:

$$\{(x,t) \mid x \in U_{h(t)}\} \rightarrowtail U_{\mathcal{E}} \times W_1 \xrightarrow{\pi_2} W_1 \ . \tag{4}$$

By definition

$$U_{\omega_1^\flat} = \bigcup_{t \in W_1} U_{h(t)} \ .$$

Next, we unwind the characterising properties of U_a and h in $\mathcal{E}$ to calculate, for a global element t of W_1, properties of $U_{h(t)}$ as a subobject of $U_{\mathcal{E}}$. For convenience, we adopt set-theoretic notation, with obvious interpretations in $\mathcal{E}$. For $0 \in W_1$, trivially

$$U_{h(0)} = \emptyset \ . \tag{5}$$

For a successor element:

$$\begin{aligned}
U_{h(s(t))} &= \bigcup_{b \in h(s(t))} \{x \mid \forall y \in x.\, y \in U_b\} \\
&= \bigcup_{b \in h(t) \cup \{h(t)\}} \{x \mid \forall y \in x.\, y \in U_b\} \\
&= \Big(\bigcup_{b \in h(t)} \{x \mid \forall y \in x.\, y \in U_b\}\Big) \cup \{x \mid \forall y \in x.\, y \in U_{h(t)}\} \\
&= U_{h(t)} \cup \{x \mid \forall y \in x.\, y \in U_{h(t)}\} \\
&= \{x \mid \forall y \in x.\, y \in U_{h(t)}\} && (6) \\
&= \{x \mid \neg \mathsf{S}(x)\} \cup \mathcal{P}_s(U_{h(t)}) \\
&\cong \mathsf{At} + \mathcal{P}_s(U_{h(t)}) \ , && (7)
\end{aligned}$$

where equality (6) holds because every U_a (in particular $U_{h(t)}$) is transitive, hence $U_{h(t)} \subseteq \{x \mid \forall y \in x.\, y \in U_{h(t)}\}$. Finally, for a "limit" element $l((t_n)_{n\in\mathbb{N}})$[5]

$$\begin{aligned}
U_{h(l((t_n)_{n\in\mathbb{N}}))} &= \bigcup_{b \in h(l((t_n)_{n\in\mathbb{N}}))} \{x \mid \forall y \in x.\, y \in U_b\} \\
&= \bigcup_{n\in\mathbb{N}} \bigcup_{b \in h(t_n)} \{x \mid \forall y \in x.\, y \in U_b\} \\
&= \bigcup_{n\in\mathbb{N}} U_{h(t_n)} \ . && (8)
\end{aligned}$$

[5]Because N is a countable copower, global elements of $(W_1)^{\mathsf{N}}$ are in one-to-one correspondence with external sequences $(t_n)_{n\in\mathbb{N}}$ of global elements of W_1.

Using the Axiom of Choice, we choose cofinal approximating sequences for limit ordinals to build a transfinite sequence $(t_\alpha)_{\alpha<\omega_1}$ of global elements of W_1 by:

$$\begin{aligned} t_0 &= 0 \\ t_{\alpha+1} &= s(t_\alpha) \\ t_\lambda &= l((t_{\alpha_n})_{n\in\mathbb{N}}) \quad \lambda<\omega_1 \text{ a limit ordinal, } \alpha_n<\lambda,\ (\alpha_n)_{n\in\mathbb{N}} \text{ cofinal in } \lambda\ . \end{aligned}$$

Using the characterisations of the various $U_{h(t)}$ as subobjects of $U_{\mathcal{E}}$ above, and the definition of $U_{\omega_1^\flat}$ as a subobject of $U_{\mathcal{E}}$, one has, by external transfinite induction on $\alpha, \beta < \omega_1$, that

$$\alpha \leq \beta \implies U_{h(t_\alpha)} \subseteq U_{h(t_\beta)} \subseteq U_{\omega_1^\flat} \tag{9}$$

and these inclusions coincide with the ones constructed in the proof of Proposition 5.17 for the case where A is $\mathsf{At}_{\mathcal{E}}$. By (9), inclusions to the object $U_{\omega_1^\flat}$ form a cocone, hence the colimiting property of $U_{\mathcal{E}}$ gives a morphism, commuting with inclusions, $c\colon U_{\mathcal{E}} \longrightarrow U_{\omega_1^\flat}$. By the colimiting property of $U_{\mathcal{E}}$, the composite of c followed by the inclusion $U_{\omega_1^\flat} \subseteq U_{\mathcal{E}}$ is the identity on $U_{\mathcal{E}}$. Thus the inclusion $U_{\omega_1^\flat} \subseteq U_{\mathcal{E}}$ is a regular epi, hence an isomorphism. That is, $U_{\omega_1^\flat} = U_{\mathcal{E}}$. □

Corollary 6.2. $\mathsf{Idl}_\infty(\mathcal{E}) \models$ *"$\omega_1^\flat$ is not a set".*

Proof. By set induction on the index a, one proves easily that $a \notin U_a$ holds for all $a \in U$. Were $\omega_1^\flat$ a set, we would get a contradiction by $\omega_1^\flat \in U = U_{\omega_1^\flat}$. □

The same argument shows that no superclass of $\omega_1^\flat$ is a set, in particular, $\omega_1^\sharp$ is not a set.

Corollary 6.3. $\mathsf{Idl}_\infty(\mathcal{E}) \models$ *"W_1 is not a set".*

Proof. Were W_1 a set then, by Replacement, its image under h would also be a set, but this is $\omega_1^\flat$. □

Aczel's *Regular Extension Axiom* (REA), [2], is an axiom that can be added to $\mathbf{CZFA_{Exp}}$ in order to ensure that inductive definitions which are *bounded* by a set (roughly, this corresponds to having a set of generators) give rise to inductively defined classes which are themselves sets. A straightforward consequence of REA is that W_1 is a set. Thus we have:

Corollary 6.4. $\mathsf{Idl}_\infty(\mathcal{E}) \models \neg$REA.

6.2. Failure of full Separation

A further consequence of Theorem 6.1 is that the full separation schema is *never* validated in $\mathsf{Idl}_\infty(\mathcal{E})$.

Theorem 6.5. $\mathsf{Idl}_\infty(\mathcal{E}) \not\models$ Sep.

Proof. We show that, in $\mathbf{CZFA_{Exp}}$ + Separation, it holds that W_1 is a set. The result then follows from Corollary 6.3.

Working in $\mathbf{CZFA_{Exp}}$, we carve out an isomorphic copy of W_1 as a subclass of $\mathsf{N}^{\mathsf{N}^{\mathsf{N}}}$. Consider the algebra structure for $0, s, l$ over $\mathsf{N}^{\mathsf{N}^{\mathsf{N}}}$ given by:

$$\begin{aligned}
0 &= \lambda f.\, 0 \\
s(F) &= \lambda f. \begin{cases} 1 & \text{if } f(0) = 0 \\ F(\lambda n. f(n{+}1)) & \text{if } f(0) = 1 \\ 0 & \text{otherwise} \end{cases} \\
l((F_n)_{n \in \mathsf{N}}) &= \lambda f. \begin{cases} 2 & \text{if } f(0) = 0 \\ F_{f(0)-1}(\lambda n. f(n{+}1)) & \text{otherwise} \end{cases}
\end{aligned}$$

By the initiality of W_1, there is a unique algebra homomorphism g from W_1 to $\mathsf{N}^{\mathsf{N}^{\mathsf{N}}}$. Consider the subclass of W_1 consisting of those t satisfying: for all $t' \in W_1$, if $g(t') = g(t)$ then $t' = t$. It is routine to prove that this is a subalgebra of W_1, hence it is the whole of W_1. In other words, g is an injective function. The image of g is thus a subclass of the set $\mathsf{N}^{\mathsf{N}^{\mathsf{N}}}$ that is isomorphic to W_1.

Finally, assuming Separation, subclasses of sets are sets. In particular, the image of g is a set, whence W_1 is a set. □

6.3. The Powerset axiom

As is well that, in the presence of the Exponentiation axiom, the Powerset axiom is equivalent to the class of subsets of $\{\emptyset\}$ forming a set.

Proposition 6.6. $\mathsf{Idl}_\infty(\mathcal{E}) \models \mathrm{Pow}$ *if and only if $\mathcal{E}$ is an elementary topos.*

Proof. The object $\mathcal{P}_s(1)$ in $\mathsf{Idl}_\infty(\mathcal{E})$ is given by the presheaf $\mathrm{Sub}_\mathcal{E}$ sending an object A of $\mathcal{E}$ to the lattice of subobjects of A in $\mathcal{E}$ and whose morphism part is given by pulling back subobjects along morphisms in $\mathcal{E}$. Thus $\mathcal{P}_s(1)$ is a set iff $\mathrm{Sub}_\mathcal{E}$ is representable iff $\mathcal{E}$ has a subobject classifier iff $\mathcal{E}$ is a topos. □

By Theorem 6.5 and Proposition 6.6, any small constructive topos with countable sums that is not a topos provides, via $\mathsf{Idl}_\infty(\mathcal{E})$, a model for $\mathbf{CZFA_{Exp}}$ in which both Separation and Powerset fail. Quite a few, mathematically natural, examples have been discussed at the end of Section 3.

6.4. Fullness

Aczel's set theory $\mathbf{CZF}$ differs from the theory $\mathbf{CZF_{Exp}}$ considered here by having, instead of the Exponentiation axiom, a schema called Subset Collection. This is strictly stronger than Exponentiation, [21], and strictly weaker than Powerset. In the presence of the other axioms, Subset Collection is equivalent to an Axiom called *Fullness*:

> for any two sets X, Y, there is a set Z of total relations between X and Y, such that any total relation contains one in Z.

In this subsection, we give sufficient conditions for Fullness to hold in $\mathsf{Idl}_\infty(\mathcal{E})$.

First we review a condition (F) on class structure corresponding to Fullness, introduced by van den Berg and Moerdijk in [7, 9, 8]. For morphisms $a : A \longrightarrow X$ and $b : B \longrightarrow X$, in a regular category $\mathcal{C}$, let $M_X(a,b)$ denote the external poset of those relations $r\colon R \rightarrowtail A \times_X B$ in $\mathcal{C}/X$ for which $\pi_1 \circ r : R \longrightarrow A$ is a regular epi. In other words $M_X(a,b)$ is the poset of *total* relations (also known as *many-valued* relations) between a and b in $\mathcal{C}/X$. Since such spans are preserved by pullbacks every morphism $f : Y \to X$ induces a monotone map $f^* : M_X(a,b) \to M_Y(f^*a, f^*b)$. In this paper, we consider van den Berg and Moerdijk's axiom in the setting of a category $\mathcal{C}$ with basic well-founded class structure $\mathcal{S}$.

(F) For any two small maps $a : A \longrightarrow X$ and $b : B \longrightarrow X$ there exist a regular epi $p : Y \twoheadrightarrow X$, a small map $c : C \longrightarrow Y$ and an $R \in M_C(c^*p^*a, c^*p^*b)$ such that, for every $d : D \longrightarrow Y$ and $S \in M_D(d^*p^*a, d^*p^*b)$, there exists a regular epi $q : E \twoheadrightarrow D$ and a map $f : E \longrightarrow C$ with $d \circ q = c \circ f$ and $f^*R \leq q^*S$.

As remarked in [9], although complicated, (F) arises naturally as the Kripke-Joyal translation of the set-theoretic Fullness property formulated above. As in [8, Proposition 7.2(4)], a category $\mathcal{C}$ with basic well-founded class structure satisfies (F) if and only if the set-theoretic Fullness axiom holds in the interpretation of the first-order language in $\mathcal{C}$.

We now give a corresponding Fullness axiom on a constructive topos $\mathcal{E}$. When $\mathcal{E}$ has countable sums, this axiom will ensure that $\mathsf{Idl}_\infty(\mathcal{E})$ satisfies condition (F).

Definition 6.7. A pretopos $\mathcal{E}$ enjoys *type-theoretic fullness* if for all $a : A \longrightarrow X$ and $b : B \longrightarrow X$ in $\mathcal{E}$ there exist a cover $p : Y \twoheadrightarrow X$, a morphism $c : C \longrightarrow Y$ and $R \in M_C(c^*p^*X, c^*p^*B)$ such that for every $d : D \longrightarrow Y$ and $S \in M_D(d^*p^*A, d^*p^*B)$ there exists a cover $q : E \twoheadrightarrow D$ and a map $f : E \longrightarrow C$ with $d \circ q = c \circ f$ and $f^*R \subseteq q^*S$.

Note that this is just condition (F) with smallness assumptions dropped. (All maps in $\mathcal{E}$ are to be thought of as small.)

Proposition 6.8. *If $\mathcal{E}$ is a constructive topos with countable sums satisfying type-theoretic smallness then $\mathsf{Idl}_\infty(\mathcal{E})$ satisfies (F), hence validates the set-theoretic Fullness axiom.*

Proof. Using the Kripke-Joyal interpretation it is straightforward, but tedious to show that type theoretic fullness for $\mathcal{E}$ guarantees that the class $\mathcal{S}_\mathcal{E}$ of representable morphisms in $\mathsf{Idl}_\infty(\mathcal{E})$ validates axiom (F). □

By interpreting the first-order language over $V_\mathcal{E}$ (rather than over $U_\mathcal{E}$), it follows that type-theoretic fullness suffices for a constructive topos $\mathcal{E}$ with countable sums to model full **CZF**.

7. Discussion

We have argued that locally cartesian closed pretoposes provide a good notion of *constructive topos*. And we have shown that any constructive topos with sufficient coproducts can be viewed as the category of sets within a model of $\mathbf{CZFA_{Exp}}$; in particular, it provides a model of the set theory $\mathbf{CZF_{Exp}}$. This fact is analogous, for constructive toposes, to Fourman and Hayashi's result that elementary toposes with small coproducts model **IZF** [14, 16].

In this paper, sufficient coproducts means countable coproducts. But this leads to models of $\mathbf{CZF_{Exp}}$ with pathological properties ($V = V_{\omega_1}$, the incompatibility with full Separation and with REA). It would be interesting to see if constructive toposes with all small coproducts give rise to less pathological models of $\mathbf{CZF_{Exp}}$.

A weakness in our presentation is that we have used **ZFC** as the metatheory to analyse $\mathsf{Idl}_\infty(\mathcal{E})$. This means that our claim, in Section 1, that the theory $\mathbf{CZF_{Exp}}$ is justified as being of interest through its wide range of natural mathematical models (constructive toposes with countable sums) is not yet as philosophically neutral as it should be. We believe that this is not a fundamental issue. With sufficient care, our dependency on classical properties of the ordinal ω_1 should be eliminable in favour of constructively acceptable proofs.

A tempting way to approach such a weakening of the meta-theory would be to avoid the use of algebraic set theory altogether, and instead to give a direct forcing-style interpretation of the language of set theory in a constructive topos with countable sums. The first author has outlined one possible such interpretation in talks on this work, but the details have not been verified. A further benefit of adopting a forcing-style approach would be that it avoids any need for assuming that the constructive topos $\mathcal{E}$ is small.

Acknowledgements

We thank Benno van den Berg for helpful comments. This paper was written while the authors were on sabbatical at the Universities of Ljubljana and Sussex respectively. We thank Andrej Bauer and Bernhard Reus for hosting these visits.

References

[1] P. Aczel. The type theoretic interpretation of constructive set theory. In *Logic Colloquium '77*, number 96 in Stud. Logic Foundations Math., pages 55–66. North Holland, 1978.

[2] P. Aczel. The type theoretic interpretation of constructive set theory: Inductive definitions. In *Logic, Methodology and Philosophy of Science VII*. North Holland, 1986.

[3] P. Aczel and M. Rathjen. Notes on constructive set theory. Technical Report 40, 2000/2001, Mittag-Leffler Institute, Sweden, 2001.

[4] S. Awodey, C. Butz, A.K. Simpson, and Th. Streicher. Relating first-order set theories, toposes and categories of classes, 2010. submitted.

[5] S. Awodey and H. Forssell. Algebraic models of intuitionistic theories of sets and classes. *Theory and applications of categories*, 15(5):147–163, 2005.

[6] S. Awodey and M. Warren. Predicative algebraic set theory. *Theory and Applications of Categories*, 15(1):1–39, 2005.

[7] B. van den Berg. *Predicative Topos Theory and Models for Constructive Set Theory*. PhD thesis, Utrecht University, 2006.

[8] B. van den Berg and I. Moerdijk. Aspects of predicative algebraic set theory I: exact completion. *Annals of Pure and Applied Logic*, 156(1):123–159, 2008.

[9] B. van den Berg and I. Moerdijk. A unified approach to algebraic set theory. In *Logic Colloquium 2006*, Lecture Notes in Logic, pages 18–37. Cambridge University Press, 2009.

[10] B. van den Berg and I. Moerdijk. Aspects of predicative algebraic set theory III: sheaves, 2010. Preprint: `http://arxiv.org/abs/0912.1242`.

[11] B. van den Berg and I. Moerdijk. Aspects of predicative algebraic set theory II: realizability. *Theoretical Computer Science*, to appear.

[12] L. Birkedal, A. Carboni, G. Rosolini, and D.S. Scott. Type theory via exact categories. In *Proc. Logic in Computer Science (LICS)*, pages 188–198, 1998.

[13] F. Borceux. *Handbook of Categorical Algebra*. Cambridge University Press, 1994. 3 volumes.

[14] M.P. Fourman. Sheaf models for set theory. *J. Pure Appl. Algebra*, 19:91–101, 1980.

[15] H. Friedman. Set theoretic foundations for constructive analysis. *Ann. of Math.*, 105:1–28., 1977.

[16] S. Hayashi. On set theories and toposes. In *Proceedings of the Logic Conference, Hakone*, volume 891 of *LNM*. Springer-Verlag, 1980.

[17] P.T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vols. 1–2*. Number 43–44 in Oxford Logic Guides. Oxford University Press, 2002. 2 volumes.

[18] A. Joyal and I. Moerdijk. *Algebraic set theory*. Number 220 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1995.

[19] F. W. Lawvere and R. Roseburgh. *Sets for Mathematicians.* Cambridge University Press, 2003.

[20] P. Lietz and Th. Streicher. Impredicativity entails untypedness. *Math. Struct. in Comp. Sci.*, 12:335–347, 2002.

[21] R. Lubarsky. Independence results around constructive ZF. *Ann. Pure Appl. Logic*, 132:209–225, 2005.

[22] M. Menni. Characterization of the left exact categories whose exact completions are toposes. *Jour. Pure Appl. Algebra*, 177:287–301, 2003.

[23] I. Moerdijk and E. Palmgren. Well-founded trees in categories. *Annals of Pure and Applied Logic*, 104:189–218, 2000.

[24] I. Moerdijk and E. Palmgren. Type theories, toposes and constructive set theory: Predicative aspects of AST. *Annals of Pure and Applied Logic*, 114:155–201, 2002.

[25] J. Myhill. Constructive set theory. *J. Symbolic Logic*, 40:347–382, 1975.

[26] J. van Oosten. *Realizability: an Introduction to its Categorical Side*, volume 152 of *Studies in Logic and the Foundations of Mathematics.* Elsevier, 2008.

[27] M.A. Shulman. Stack semantics and the comparison of material and structural set theories, 2010. Preprint: `http://arxiv.org/pdf/1004.3802v1`.

[28] A.K. Simpson. Elementary axioms for categories of classes. In *Proc. 14th Annual IEEE Symposium on Logic in Computer Science*, pages 77–85, 1999.